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Shifted 1/*N* expansion for the Dirac equation for vector and scalar potentials

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Abstract. A relativistic shifted $1/N$ expansion method has been applied to get the energy values of the Dirac equation with a potential having both scalar and vector components. The analytical results obtained for the energy values are applicable for any potential of the above form and the results have been compared with the numerical results of some realistic power law quark confining potentials having both scalar and vector interactions.

A new approximate analytical technique known as the shifted $1/N$ expansion (Imbo *et al* 1984) has received increased attention recently in solving the N-dimensional stationary Schrödinger equation where N is the space dimension. Since it leads to algebraic equations which can be handled easily with highly accurate results, it has been applied to a large number of physically interesting potentials (Roychoudhury and Varshni 1988b, c, Roy *et al* 1988; for other references see Papp 1988).

Until very recently, however, the use of this method has been restricted to non-relativistic problems. Some time ago, Miramontes and Pajares (1984) studied the large-N limit of both the Klein–Gordon and Dirac equations. But their result is not of much practical use as they dealt with a pure Coulomb problem which is exactly solvable (see also Panja and Dutt 1988).

Recently, Roychoudhury and Varshni (1987) developed the relativistic shifted $1/N$ expansion method and used it to the linear scalar potential. They have also extended their formalism to the pure vector potential suitable for screened Coulomb-type problems (Roychoudhury and Varshni 1988a). Our objective here is to apply the method to relativistic problems and to obtain explicit analytical formulae for energy values of any radially symmetric potential having vector and scalar parts. To the best of our knowledge no work exists on the relativistic shifted $1/N$ expansion on these types of potential. Our study will be limited to the cases where the rest energy can be assumed to be large compared with the relativistic corrections.

Unlike the non-relativistic case, numerical calculations of eigenvalues for relativistic potentials are not readily available. Therefore for numerical comparison, we have considered the Dirac equation with an equally mixed 4-vector and scalar power-law potential of the form $V(r)=Ar^{\nu}+V_0$ (Jena and Tripati 1983) with $\nu=0.1$ and $A>0$ as detailed numerical results are available for this potential.

Now the Dirac equation for a potential with a vector component V and a scalar component V_2 is (Fock 1978)

$$(W-V)\psi(r)=(\alpha\cdot p)\psi(r)+(m+V_2)\beta\psi(r) \tag{1}$$

where $\psi(r)$ is a four-component wavefunction and W is the relativistic energy (in the units $\hbar = c = 1$).

The Dirac equation (1) in N dimensions can be separated in spherical polar coordinates thereby reducing to a system of two coupled differential equations for the radial wavefunctions $F(r)$ and $G(r)$ (Roychoudhury and Varshni 1987)

$$\frac{\mathrm{d}F}{\mathrm{d}r} - \frac{\chi}{4} F = (W - V + m + V_2)G \tag{2}$$

$$\frac{\mathrm{d}G}{\mathrm{d}r} + \frac{\chi}{r} G = (W - V + m + V_2)F \tag{3}$$

where we have used the standard spinors.

$$\psi_{jS}^{\mathrm{M}} = \frac{1}{r^{(N-1)/2}} \begin{bmatrix} F(r) & Y_{lj}^{\mathrm{M}} \\ \mathrm{i}G(r) & Y_{l'j}^{\mathrm{M}} \end{bmatrix} \qquad \begin{matrix} l = j + \frac{1}{2}S \\ l' = j - \frac{1}{2}S \end{matrix} \qquad S = \pm 1 \tag{4a}$$

and

$$\chi = \frac{(N_j - 2)S}{2} \qquad \text{and} \qquad N_j = N + 2j. \tag{4b}$$

When $S = +1$, $N = 3$, $j = l - \frac{1}{2}$, $\chi = l$ and when $S = -1$, $N = 3$, $j = l + \frac{1}{2}$, $\chi = -(l+1)$, (2) and (3) reduce to the usual set of Dirac equations for a radially symmetric potential. Now eliminating F and writing $W = E + m$, we get from (3) and (4)

$$\frac{\mathrm{d}^2 G}{\mathrm{d}r^2} - \frac{\chi(\chi+1)}{r^2} G + (E - \bar{V})^2 G + 2(m + V_2)(E - \bar{V})G$$
$$= \frac{1}{E - \bar{V} + 2m + 2V_2} \left(2\frac{\mathrm{d}V_2}{\mathrm{d}r} - \frac{\mathrm{d}\bar{V}}{\mathrm{d}r} \right) \left(\frac{\mathrm{d}G}{\mathrm{d}r} + \frac{\chi}{r} G \right) \tag{5}$$

where for convenience we have written $\bar{V} = V + V_2$. If we put

$$G = \left(1 + \frac{E - \bar{V} + 2V_2}{2m} \right)^{1/2} \varphi(r) \tag{6}$$

then equation (5) will be

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}r^2} - \frac{\chi(\chi-1)}{r^2} \varphi(r) + 2(m + V_2)(E - \bar{V})\varphi(r) + (E - \bar{V})^2 \varphi(r)$$
$$= \frac{1}{2m + 2V_2 + E - \bar{V}} \left(\frac{1}{2} \frac{\mathrm{d}^2\bar{V}}{\mathrm{d}r^2} + \frac{\chi}{4} \frac{\mathrm{d}\bar{V}}{\mathrm{d}r} \right) + \frac{3}{4} \frac{1}{(2m + E - \bar{V} + 2V_2)^2}$$
$$\times \left(\frac{\mathrm{d}V_2}{\mathrm{d}r} - \frac{\mathrm{d}\bar{V}}{\mathrm{d}r} \right)^2 - \frac{1}{E - \bar{V} + 2m + 2V_2} \left(\frac{\mathrm{d}^2 \bar{V}_2}{\mathrm{d}r^2} + \frac{2\chi}{4} \frac{\mathrm{d}V_2}{\mathrm{d}r} \right) \varphi(r). \tag{7}$$

Now because we are interested in problems where the rest energy is large, we can expand terms like $(2m + 2V_2 + E - \bar{V})^{-1}$ as

$$\frac{1}{2m} \left\{ 1 - \frac{E - \bar{V} + 2V_2}{2m} + \ldots \right\}.$$

Moreover, since the contributions of the terms of the order of $1/m^3$ are small we can treat them as perturbations and calculate them after the leading-order calculations have been done. Thus neglecting terms of the order of $\sim 16m^3$ we get from (7)

$$-\frac{1}{2m}\frac{\mathrm{d}^2\varphi}{\mathrm{d}r^2}+\frac{(k-1)(k-3)}{8mr^2}\varphi-\left(1+\frac{V_2}{m}\right)(E-\bar{V})\varphi=-u(r)\varphi+\frac{(E-\bar{V})^2}{2m}\varphi \tag{8}$$

where

$$u(r)=\frac{1}{4m^2}\left\{\frac{1}{2}\frac{\mathrm{d}^2\bar{V}}{\mathrm{d}r^2}+\frac{\chi}{4}\left(\frac{\mathrm{d}\bar{V}}{\mathrm{d}r}-2\frac{\mathrm{d}V_2}{\mathrm{d}r}\right)-\frac{\mathrm{d}^2V_2}{\mathrm{d}r^2}\right\} \tag{9}$$

and

$$k=\begin{cases}N+2j+1 & \text{when} \quad j=l-\frac{1}{2}\\ N+2j-1 & \text{when} \quad j=l+\frac{1}{2}\end{cases} \tag{10a}$$

so that we can always write

$$k=N+2l \tag{10b}$$

in conformity with the non-relativistic case and also

$$\chi=\begin{cases}\dfrac{k-3}{2} & \text{when} \quad j=l-\frac{1}{2}\\ -\dfrac{(k-1)}{2} & \text{when} \quad j=l+\frac{1}{2}.\end{cases} \tag{10c}$$

Let us now give a shift to the quantity k, i.e.

$$k=\bar{k}+a \tag{11}$$

Then equation (8) can be written as

$$-\frac{1}{2m}\frac{\mathrm{d}^2\varphi}{\mathrm{d}r^2}+\frac{\bar{k}^2}{8mr^2}\left(1+\frac{a-1}{\bar{k}}\right)\left(1+\frac{a-3}{\bar{k}}\right)\varphi+\left(\bar{V}-\frac{\bar{V}^2}{2m}+\frac{E\bar{V}}{m}\right)\varphi$$
$$+u(r)\varphi-\frac{V_2}{m}(E-\bar{V})\varphi=\left(E+\frac{E^2}{2m}\right)\varphi \tag{12}$$

where χ in $u(r)$ is given by

$$\chi=\begin{cases}-\dfrac{(\bar{k}+a-1)}{2} & \text{when} \quad j=l+\frac{1}{2}\\ \dfrac{(\bar{k}+a-3)}{2} & \text{when} \quad j=l-\frac{1}{2}.\end{cases} \tag{13}$$

It is convenient to shift the origin of coordinate to $r=r_0$ by defining (Imbo *et al* 1984)

$$x=\frac{\bar{k}^{1/2}}{r_0}(r-r_0) \tag{14}$$

and accordingly we expand $\bar{V}(r)$, $u(r)$ and E as

$$\bar{V}(r)=\frac{\bar{k}^2}{Q}\left[\bar{V}(r_0)+gxr_0\bar{V}'(r_0)+\frac{g^2}{2}x^2r_0^2\bar{V}''(r_0)+\ldots\right] \tag{15}$$

$$u(r)=\frac{\bar{k}^2}{Q}\left[u(r_0)+gxr_0u'(r_0)+\frac{g^2}{2}x^2r_0^2u''(r_0)+\ldots\right] \tag{16}$$

$$E=\bar{k}^2E_0+\bar{k}E_1+E_2+\frac{E_3}{\bar{k}}+\ldots \tag{17}$$

where Q is a scale, whose magnitude is to be determined later, and

$$g=1/k. \tag{18}$$

After substituting equations (15)-(17) in equation (12), the leading-order energy term is given by the equation

$$\frac{\bar{k}}{8m}+\frac{r_0^2}{\bar{k}}\bar{V}(r_0)-\frac{r_0^2}{2m\bar{k}}\bar{V}(r_0)^2+\frac{r_0^2}{m\bar{k}}\bar{V}(r_0)E^{(0)}+\frac{r_0^2}{\bar{k}}u(r_0)-\frac{r_0^2}{\bar{k}m}(v_2(r_0)E^{(0)}-V_2(r_0)\bar{V}(r_0))$$
$$=\frac{r_0^2}{\bar{k}}E^{(0)}+\frac{r_0^2}{2m\bar{k}}E^{(0)^2}. \tag{19}$$

In the above derivations Q is chosen to be $\bar{k}^2$ which gives back the correct Dirac equation for any N. Equation (19) gives

$$E_0=\bar{V}(r_0)-m-V_2(r_0)+m\left\{1+\frac{V_2(r_0)^2}{m^2}+\frac{2V_2(r_0)}{m}+\frac{\bar{U}(r_0)}{2m^3}+\frac{\bar{k}^2}{4m^2r_0^2}\right\}^{1/2} \tag{20}$$

where

$$\bar{U}(r)=\frac{1}{2}\frac{\mathrm{d}^2\bar{V}}{\mathrm{d}r^2}+\frac{\chi}{4}\left(\frac{\mathrm{d}\bar{V}}{\mathrm{d}r}-2\frac{\mathrm{d}V_2}{\mathrm{d}r}\right)-\frac{\mathrm{d}^2V_2}{\mathrm{d}r^2}. \tag{21}$$

Now r_0 is chosen so as to make E_0 a minimum, i.e.

$$\frac{\mathrm{d}E_0}{\mathrm{d}r_0}=0 \qquad \frac{\mathrm{d}^2E_0}{\mathrm{d}r_0^2}>0. \tag{22}$$

Equation (22) gives

$$\bar{k}^2=\frac{r_0^3\bar{U}'(r_0)}{m}+4mr_0^3V_2'(r_0)\left(1+\frac{V_2(r_0)}{m}\right)+2r_0^4(\bar{V}'(r_0)-V_2'(r_0))^2$$
$$+2mr_0^3(V_2'(r_0)-\bar{V}'(r_0))\left\{4\left(1+\frac{V_2(r_0)}{m}\right)^2+\frac{2}{m^3}\bar{U}(r_0)\right.$$
$$\left.+\frac{\bar{U}'(r_0)r_0}{m^3}+\frac{4r_0}{m}V_2'(r_0)\left(1+\frac{V_2(r_0)}{m}\right)+\frac{r_0^2}{m^2}(\bar{V}'(r_0)-V_2'(r_0))^2\right\}^{1/2}. \tag{23}$$

The terms of the order of $\bar{k}$, $1/\bar{k}$ etc can be obtained in exactly the same manner as was done in the case of Schrödinger equation (Imbo *et al* 1984). We here quote the results and incorporate the appropriate modifications whenever applicable.

The next contribution to E is of order $\bar{k}$ and is given by

$$\frac{\bar{k}}{r_0^2}\left[(n_4+\tfrac{1}{2})w-\frac{(2-a)}{4m}\right] \tag{24}$$

where

$$w=\frac{1}{2m}\left[3+\frac{4mr_0^4}{Q}X''\right] \tag{25}$$

where

$$X''=\bar{V}''(r_0)-\frac{\bar{V}'(r_0)^2}{m}-\frac{\bar{V}(r_0)\bar{V}''(r_0)}{m}+\frac{\bar{V}''(r_0)E^{(0)}}{m}+\frac{\bar{U}''(r_0)}{4m^2}-\frac{V_2''(r_0)E^{(0)}}{m}$$
$$+\frac{\bar{V}(r_0)V_2''(r_0)}{m}+\frac{2V_2'(r_0)\bar{V}'(r_0)}{m}+\frac{V_2(r_0)\bar{V}''(r_0)}{m}.$$

In analogy with the non-relativistic case the shift a is chosen so as to make contribution (24) vanish. (It has been shown by Roychoudhury and Varshni (1988) that this coice gives the correct result for the relativistic hydrogen atom up to the appropriate order in $1/m$). Now the condition that the expression (24) would give zero contribution is

$$a=2-2(2n_4+1)mw. \tag{26}$$

Now equation (11) together with equations (10b) and (26) gives the required equation for r_0 as

$$N+2l-2+(2n_r+1)2mw=\bar{k} \tag{27}$$

The energy including the second-order correction is given by,

$$E=E_0+E_2 \tag{28}$$

where E_0 is given by equation (20) and

$$E_2=-F_2\left(1+\frac{E^{(0)}+V_2(r_0)-\bar{V}(r_0)}{m}\right)^{-1} \tag{29}$$

where

$$F_2=\lambda/r_0^2 \tag{30}$$

and

$$\lambda=(1/8m)(1-a)(3-a)+(1+2n_r)\bar{\varepsilon}_2+3(1+2n_r+2n_r^2)\bar{\varepsilon}_4$$
$$-\frac{1}{w}[\bar{\varepsilon}_1^2+6(1+2np)\bar{\varepsilon}_1\bar{\varepsilon}_3+(11+30n_r+30n_r^2)\bar{\varepsilon}_3^2] \tag{31}$$

where

$$\bar{\varepsilon}_j=\frac{\xi_j}{(2mw)^{j/2}} \qquad j=1,2,\ldots \tag{32}$$

and

$$\varepsilon_1=\frac{(2-a)}{2m} \qquad \varepsilon_2=-\frac{3(2-a)}{4m} \tag{33}$$

$$\varepsilon_3=-\frac{1}{2m}+\frac{r_0^5}{6Q}\left(\frac{\bar{V}'(r_0)\bar{V}''(r_0)}{m}-\frac{\bar{V}(r_0)\bar{V}'''(r_0)}{m}+\frac{\bar{V}'''(r_0)E^{(0)}}{m}\right.$$
$$\left.+\frac{\bar{U}'''(r_0)}{4m^2}-\frac{V_2'''(r_0)E^{(0)}}{m}\right) \tag{34}$$

$$\varepsilon_4=\frac{5}{8m}+\frac{r_0^6}{24Q}\left(\bar{V}^{IV}(r_0)-\frac{3\bar{V}''(r_0)^2}{m}-\frac{\bar{V}(r_0)\bar{V}^{IV}(r_0)}{m}-\frac{4\bar{V}'(r_0)\bar{V}'''(r_0)}{m}\right.$$
$$\left.+\frac{\bar{V}^{IV}(r_0)E^{(p)}}{m}+\frac{\bar{U}^{IV}(r_0)}{4m^2}-\frac{V_2^{IV}(r_0)E^{(0)}}{m}\right). \qquad (35)$$

We first checked our formulae for scalar potential of the form λr and found that our formulae reproduced the numerical results of Roychoudhuri and Varshni (1987). It should be mentioned that these authors treated the Dirac equation with scalar potential essentially as a Schrödinger equation with the potential duly modified.

Next we applied our formalism for the Dirac equation with an equally mixed 4-vector and scalar power law potential of the form $V(r)=Ar^{0.1}+V_0$ which reproduces the most recent data of ψ and γ spectroscopies (Martin 1980, 1981, Barik and Jena 1980, 1982, Khare 1981). Jena and Tripati (1983) extended this model to fit the mass spectra of $Q\bar{Q}$, $q\bar{q}$ and $Q\bar{q}$ systems in a unified manner. This model has the advantage that the Dirac equation can then be reduced to a Schrödinger equation and exact numerical solution of the eigenvalue can be found. In table 1 we compare the eigenvalues ε_{nl} (corresponding to the confined bound state of quarks) related to the Dirac quark binding energy W by the following relation:

$$\varepsilon_{nl}=(W-m_q-2V_0)[(W+m_q)(2A)^{-2/\nu}]^{\nu/(\nu+2)}$$

when the quark mass $m_q=1.709$, $V_0=-2.028$ and $A=1.8031$. We have neglected the second-order correction whenever it is greater than 5%. Because it was pointed out by Maluendes *et al* (1986, 1987) that whenever the lowest-order calculation gives a better result than those with higher-order corrections, one encounters the divergence problem of perturbation series and then one should use a modified version (Maluendes *et al* 1986, 1987) of the higher-order corrections for the shifted $1/N$ expansion to ensure better convergence. As can be seen from the table the agreement of our results with the numerical ones is excellent, the maximum error being 2.7%, though the shifted $1/N$ expansion is not expected to yield very accurate results for the lowest lying state especially when the radial quantum number is large.

To conclude, we have developed a general formalism for the shifted $1/N$ expansion of the Dirac equation with potential having both vector and scalar components and this formalism gives very good results even to the lowest order.

Table 1. Spin-averaged eigenvalues ε_{nl} for the potential whose vector and scalar components are identical and are given by $V_V=V_S=Ar^\nu+V_0$.

State	ε_{nl} †	ε_{nl} ‡
1S	1.2364	1.240
2S	1.3347	1.340
3S	1.3923	1.398
4S	1.4335	1.439
5S	1.4657	1.471
2P	1.3071	1.309§
3P	1.3731	1.411§
3D	1.3544	1.358§

† Jena and Tripati 1983.
‡ Present calculation.
§ Denotes the values taking second-order correction.

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